Journal of Multivariate Analysis **65**, 36–57 (1998)
Article No. MV971726

Elliptical Functional Models

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Received March 26, 1996; revised February 12, 1997

In this paper, functional models with not replications are investigated within the class of the elliptical distributions. Emphasis is placed on the special case of the Student-t distribution. Main results encompasses consistency and asymptotic normality of the maximum likelihood estimators. Due to the presence of incidental parameters, standard maximum likelihood methodology cannot be used to obtain the main results, which require extensions of some existing results related to elliptical distributions. Asymptotic relative efficiencies are reported which show that the generalized least squares estimator can be highly inefficient when compared with the maximum likelihood estimator under nonnormality. © 1998 Academic Press

AMS 1990 subject classifications: 62F05, 62J05.

Key words and phrases: asymptotic relative efficiency, elliptical distributions, incidental parameters, maximum likelihood estimators, Student-t distribution.

1. INTRODUCTION

Most of the studies reported in the literature about inference on functional models are related to the normal distributions. Among others, important references are Sprent (1966), Patefield (1976), Mak (1982), Gleser (1985), and Cheng and Van Ness (1991). As is well known, the p-variate normal distribution is a member of the p-variate elliptical family of distributions which is denoted by $El_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; f)$ (Fang *et al.*, 1990), where $\boldsymbol{\mu}$ is the location vector, $\boldsymbol{\Sigma}$ is a positive definite dispersion matrix, and density has the form

$$|\boldsymbol{\Sigma}|^{-1/2} f((\mathbf{z}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\mathbf{z}, \boldsymbol{\mu})), \tag{1.1}$$

$\mathbf{z} \in \mathscr{R}^p$, for some function $f(u) \geqslant 0$, $u \geqslant 0$. In this paper, we consider the multi-univariate model

$$\mathbf{Y}_i = \boldsymbol{\alpha} + \boldsymbol{\beta} x_i + \mathbf{e}_i, \tag{1.2}$$

where x, is an unobservable random variable with

$$X_i = x_i + u_i,$$

0047-259X/98 $25.00

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which we write as

$$\mathbf{Z}_i = \mathbf{a} + \mathbf{b}x_i + \boldsymbol{\varepsilon}_i, \tag{1.3}$$

where $\mathbf{Z}_i = (\mathbf{Y}_i', X_i)'$, which are observable, $\boldsymbol{\beta} = (\beta_1, ..., \beta_q)'$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_q)'$, $\mathbf{a} = (\boldsymbol{\alpha}', 0)'$ $\mathbf{b} = (\boldsymbol{\beta}', 1)'$, $\boldsymbol{\varepsilon}_i = (\mathbf{e}_i', u_i)'$, $i = 1, ..., n$, are independent and identically distributed (i.i.d.), with $\boldsymbol{\varepsilon}_1 \sim El_p(\mathbf{0}, \mathbf{I}_p; f)$, and, $p = q + 1$. Thus, standard properties of elliptical distributions imply that

$$\mathbf{Z}_i \sim El_p(\mathbf{a} + \mathbf{b}x_i, \mathbf{I}_p; f); \tag{1.4}$$

that is, $\mathbf{Z}_i$ has a density given by

$$f_i(\mathbf{z}_i; \boldsymbol{\theta}, x_i) = f(\|\mathbf{z}_i - \mathbf{a} - \mathbf{b}x_i\|^2), \tag{1.5}$$

$i = 1, ..., n$, where $\boldsymbol{\theta} = (\boldsymbol{\alpha}', \boldsymbol{\beta}')'$ and $\|\mathbf{t}\|^2 = \mathbf{t}'\mathbf{t}$, $\mathbf{t} \in \mathfrak{R}^p$.

Model (1.3) under normality has been considered by Kubokawa and Robert (1994). See also Lieftinck-Koeijers (1988). Both papers consider calibration studies in the ordinary (nonmeasurement errors present) regression setting. It has been considered more recently by Kimura (1992) in the context of comparative calibration. See also Bolfarine and Galea-Rojas (1995). Notice that the model specified by (1.4) implies that the dispersion matrix of $\boldsymbol{\varepsilon}_1$ is considered to be the identity $p \times p$ matrix. As in Mak (1982), a more general model can be obtained by considering a general known (positive definite) dispersion matrix. However, this more general situation can be reduced to the situation specified in (1.2)–(1.4) by transforming the original variables. The variables x_i, $i = 1, ..., n$, are considered to be fixed parameters, or incidental parameters, since their number increase with the sample size. Patefield (1976) has shown in the case of normal models with $q = 1$ that the asymptotic covariance matrix of the maximum likelihood estimators of α and β does not coincide with the inverse of the Fisher information matrix corresponding to the parameters α and β. Thus, alternative approaches have to be pursued to obtain the asymptotic covariance matrix in such models. The approach pursued in this paper consists in replacing x_i by "estimates" $\tilde{x}_i = \tilde{x}_i(\mathbf{z}_i, \boldsymbol{\theta})$, $i = 1, ..., n$, which can be obtained by maximizing (1.5) with respect to x_i, $i = 1, ..., n$, for fixed $\boldsymbol{\theta}$. General conditions under which the maximum likelihood estimator of $\boldsymbol{\theta}$ is consistent and asymptotically normal are established in Mak (1982). Following Mak (1982), we establish conditions under the model (1.4) so that the maximum likelihood estimators are consistent and asymptotically normal. An explicit expression is obtained for the asymptotic covariance matrix of the maximum likelihood estimators, which allows obtaining asymptotic relative efficiencies of the maximum likelihood estimators with respect to the generalized least squares estimator. As the studies show, those estimators can be highly inefficient under nonnormality. A special case of the elliptical model (1.1) is the Student-t distribution with v degrees of freedom, in which case, $\boldsymbol{\varepsilon}_1 \sim El_p(\mathbf{0}, \mathbf{I}_p; f)$, where the density f is such that

$$f(u) = k(p, v)\, v^{v/2}(v+u)^{(v+p)/2}, \qquad u \geqslant 0, \tag{1.6}$$

where $k(p, v) = \Gamma[(v+p)/2]/\Gamma[v/2]\, \pi^{p/2}$ is the normalizing constant.

In Section 2 the notation and some preliminary results due mainly to Mak (1982) are presented. Section 3 is devoted to the elliptical functional models. Using the results presented in Section 2, general conditions are established under which the maximum likelihood estimators are consistent and asymptotically normal. An expression is obtained for the asymptotic relative efficiency of the maximum likelihood estimator with respect to the generalized least squares estimator. Section 4 is dedicated to the Student-t distribution. Asymptotic normality and consistency of the maximum likelihood estimator are established and the asymptotic relative efficiency of the generalized least squares estimator is derived. As shown, the generalized least squares estimator can be highly inefficient, specially for small degrees of freedom. Several properties (which extends existing ones) of the elliptical distributions required to prove the main results are considered in the Appendix.

2. NOTATION AND PRELIMINARY RESULTS

Let $\mathbf{Z}_1, ..., \mathbf{Z}_n$, independent p-dimensional random vectors with log-likelihood function given by

$$\sum_{i=1}^{n} \log f_i(\mathbf{z}_i; \boldsymbol{\theta}, x_i), \tag{2.1}$$

where $f_i(\mathbf{z}_i; \boldsymbol{\theta}, x_i)$ is the density of $\mathbf{Z}_i$, $i = 1, ..., n$, $\boldsymbol{\theta} = (\theta_1, ..., \theta_p)' \in \boldsymbol{\Theta} \subset \mathcal{R}^p$, and $x_i \in \mathcal{X}_i \subset \mathcal{R}$, $i = 1, ..., n$, are the incidental parameters. Suppose that $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$ and $x_{i0} \in \mathcal{X}_i$, $i = 1, ..., n$, where $\boldsymbol{\theta}_0$ and $x_{10}, ..., x_{n0}$, denote the true parameter values. The expected values are taken with respect to $\boldsymbol{\theta}_0$ and x_{i0}, $i = 1, ..., n$, which will be denoted by $E_0[\,\cdot\,] = E[\,\cdot \mid \boldsymbol{\theta}_0, x_{10}, ..., x_{n0}]$. For each i and given $\boldsymbol{\theta}$, let $\tilde{x}_i = \tilde{x}_i(\mathbf{Z}_i, \boldsymbol{\theta})$, be an estimator (possibly depending on $\boldsymbol{\theta}$) of x_i, with a possibility being the conditional maximum likelihood estimator, obtained by maximizing (2.1) with respect to x_i for fixed $\boldsymbol{\theta}$. Thus, replacing x_i by $\tilde{x}_i$ in (2.1) we obtain

$$\sum_{i=1}^{n} \log f_i(\mathbf{z}_i; \boldsymbol{\theta}, \tilde{x}_i) = \sum_{i=1}^{n} h_i(\mathbf{z}_i; \boldsymbol{\theta}). \tag{2.2}$$

We also define the functions

$$q_{i\theta_j}(\mathbf{z}_i;\boldsymbol{\theta})=\frac{\partial h_i(\mathbf{z}_i;\boldsymbol{\theta})}{\partial\theta_j},\qquad j=1,...,p, \tag{2.3}$$

$$q_{i\theta_k\theta_j}(\mathbf{z}_i;\boldsymbol{\theta})=\frac{\partial^2 h_i(\mathbf{z}_i:\boldsymbol{\theta})}{\partial\theta_j\,\partial\theta_k},\qquad j,k=1,...,p, \tag{2.4}$$

and

$$q_{i\theta_k,\,\theta_j}(\mathbf{z}_i;\boldsymbol{\theta})=q_{i\theta_k}(\mathbf{z}_i;\boldsymbol{\theta})\,q_{i\theta_j}(\mathbf{z}_i;\boldsymbol{\theta}),\qquad j,k=1,...,p. \tag{2.5}$$

Moreover, let $E_0[\mathbf{A}_n,(\boldsymbol{\theta})]$ be the $p\times p$ random matrix with entry (j,k) given by

$$n^{-1}\sum_{i=1}^{n}E_0[q_{i\theta_k\theta_j}(\mathbf{Z}_i;\boldsymbol{\theta})],\qquad j,k=1,...,p. \tag{2.6}$$

In Mak (1982, Section 2) general conditions are established under which (2.2) has a maximum $\tilde{\boldsymbol{\theta}}_n=\tilde{\boldsymbol{\theta}}(\mathbf{z}_1,...,\mathbf{z}_n)$, which converges in probability to $\boldsymbol{\theta}_1$ in the interior of $\boldsymbol{\Theta}$, where $\boldsymbol{\theta}_1$ maximizes the function

$$\bar{\psi}(\boldsymbol{\theta})=n^{-1}\sum_{i=1}^{n}E_0[h_i(\mathbf{Z}_i;\boldsymbol{\theta})], \tag{2.7}$$

and

$$\sqrt{n}(\mathbf{V}_n(\boldsymbol{\theta}_1))^{-1/2}\,(E_0[\mathbf{A}_n(\boldsymbol{\theta}_1)])(\tilde{\boldsymbol{\theta}}_n-\boldsymbol{\theta}_1)\xrightarrow{d}N_p(\mathbf{0},\mathbf{I}_p), \tag{2.8}$$

where "$\xrightarrow{d}$" means convergence in distribution, with

$$\mathbf{V}_n(\boldsymbol{\theta})=n^{-1}\sum_{i=1}^{n}\mathrm{Cov}\left[\frac{\partial h_i(\mathbf{Z}_i;\boldsymbol{\theta})}{\partial\theta_j},\frac{\partial h_i(\mathbf{Z}_i;\boldsymbol{\theta})}{\partial\theta_k}\right], \tag{2.9}$$

a $p\times p$ matrix, where, as pointed out before, the expected values are taken with respect to the true values $\boldsymbol{\theta}_0$ and x_{i0}, $i=1,...,n$. It is also noted in Mak (1982) that in some situations it is possible to obtain estimators $\tilde{x}_i$ so that $\boldsymbol{\theta}_1$ depends only on $\boldsymbol{\theta}_0$ (is independent of x_{i0}); that is, there exists a function $g(\cdot)$ such that $\boldsymbol{\theta}_1=g(\boldsymbol{\theta}_0)$. If g is one to one then, a consistent estimator of $\boldsymbol{\theta}_0$ is given by $\hat{\boldsymbol{\theta}}_n=g^{-1}(\tilde{\boldsymbol{\theta}}_n)$.

3. THE ELLIPTICAL FUNCTIONAL MODEL

In this section, asymptotic properties of the maximum likelihood estimator of the structural parameter $\boldsymbol{\theta}$ are studied under the elliptical functional model defined in (1.2)–(1.4), where the true value of $\boldsymbol{\theta}$ is denoted by $\boldsymbol{\theta}_0 = (\boldsymbol{\alpha}_0', \boldsymbol{\beta}_0')'$. From (1.5) it follows that the log-likelihood function corresponding to the observed $\mathbf{z}_1, ..., \mathbf{z}_n$, is given by

$$\sum_{i=1}^{n} \log f_i(\mathbf{z}_i; \boldsymbol{\theta}, x_i) = \sum_{i=1}^{n} \log f(\|\mathbf{z}_i - \mathbf{a} - \mathbf{b}x_i\|^2), \tag{3.1}$$

for some function $f(u)$, $u \geqslant 0$ such that $\int_0^\infty r^{p-1} f(r^2)\, dr < \infty$, which guarantees that $f(\mathbf{x}'\mathbf{x})$, $\mathbf{x} \in \mathscr{R}^p$ is a p-dimensional spherical density. Moreover, suppose that the function f satisfies the following conditions:

(C.1) $f \in \mathscr{C}^{(2)}$ and is decreasing in $(0, \infty)$;

(C.2) $\int_0^\infty r^{p+3} f(r^2)\, dr < \infty$, which guarantees finite fourth moments.

The following notation will be used in the sequel. For any function (can be a matrix) $\boldsymbol{\Phi} = \boldsymbol{\Phi}(\boldsymbol{\theta})$ we denote by $\boldsymbol{\Phi}_0$ the function evaluated at $\boldsymbol{\theta}_0$, that is, $\boldsymbol{\Phi}_0 = \boldsymbol{\Phi}(\boldsymbol{\theta}_0)$. Moreover, let

$$\mathbf{b}_1 = \frac{\mathbf{b}}{\|\mathbf{b}\|}, \qquad \mathbf{B}_1 = \mathbf{b}_1 \mathbf{b}_1', \qquad \mathbf{B}_2 = \mathbf{I}_p - \mathbf{B}_1,$$

and consider the random vectors

$$\mathbf{R}_i = \mathbf{B}_1(\mathbf{Z}_i - \mathbf{a} - \mathbf{b}x_{i0}), \qquad r_i = \mathbf{b}_1' \mathbf{R}_i, \tag{3.2}$$

and

$$\mathbf{T}_i = \mathbf{B}_2(\mathbf{Z}_i - \mathbf{a} - \mathbf{b}x_{i0}) = \mathbf{B}_2(\mathbf{Z}_i - \mathbf{a}). \tag{3.3}$$

Note that $\mathbf{R}_i = \mathbf{R}_i(\boldsymbol{\theta})$ and $\mathbf{T}_i = \mathbf{T}_i(\boldsymbol{\theta})$, $i = 1, ..., n$, are orthogonal for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and, according to Lemma A.1(i), it follows that

$$(\mathbf{T}_i', \mathbf{R}_i')' \overset{ind}{\sim} El_{2p}(\mathbf{C}[(\mathbf{a}_0 - \mathbf{a}) + (\mathbf{b}_0 - \mathbf{b})\, x_{i0}], \mathbf{C}\mathbf{C}'),$$

with $\mathbf{C} = [\mathbf{B}_2', \mathbf{B}_1']'$, which is a singular elliptical distribution. Particularly, for $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ it follows that

$$(\mathbf{T}_{i0}', \mathbf{R}_{i0}')' \overset{iid}{\sim} El_{2p}(\mathbf{0}, \mathbf{C}_0\mathbf{C}_0'),$$

from where it follows (see Lemma A.1(ii)) that

$$\begin{pmatrix} \mathbf{T}_{i0} \\ \mathbf{R}_{i0} \end{pmatrix} \overset{d}{=} \begin{pmatrix} \boldsymbol{\Gamma}'\mathbf{Q}_1'\mathbf{e}_1 \\ \boldsymbol{\Gamma}'\mathbf{Q}_2' u_1 \end{pmatrix}, \tag{3.4}$$

where $(\mathbf{e}_1', u_1)' \sim El_p(\mathbf{0}, \mathbf{I}_p; f)$, and $\boldsymbol{\Gamma}$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ are as defined in Lemma A.1 with $p_1 = q$, $p_2 = 1$, and "$v \overset{d}{=} w$" meaning that v and w are identically distributed. The conditional maximum likelihood estimator of x_i, $i=1, ..., n$, is considered next. The proof follows directly from (3.1) and (2.2).

LEMMA 3.1. *Consider the model defined by* (1.2)–(1.4). *Under condition* (C.1), *the conditional maximum likelihood estimator of* x_i *given* $\boldsymbol{\theta}$ *can be written as*

$$\tilde{x}_i = \frac{\mathbf{b}'}{\|\mathbf{b}\|^2}(\mathbf{Z}_i - \mathbf{a}),$$

$i=1, ..., n$, *and*

$$h_i(\mathbf{z}_i; \boldsymbol{\theta}) = \log f(\|\mathbf{T}_i\|^2), \tag{3.5}$$

where $\mathbf{T}_i$ *is as defined in* (3.3), $i=1, ..., n$.

Considering $\boldsymbol{\theta} = (\boldsymbol{\alpha}', \boldsymbol{\beta}') = (\alpha_1, ..., \alpha_q, \beta_1, ..., \beta_q)' \in \boldsymbol{\Theta}$, $W_f(u) = f'(u)/f(u)$, and $\Delta_f(u) = W_g(u)\, W_f(u) - W_f(u)^2$, where $g = f'$, $u \geqslant 0$, it follows from (2.3), (2.4), (2.5), and (3.5) that

$$q_{i\theta_k}(\mathbf{z}_i; \boldsymbol{\theta}) = W_f(\|\mathbf{T}_i\|^2)\,\frac{\partial\,\|\mathbf{T}_i\|^2}{\partial\theta_k}, \qquad k=1, ..., 2q, \tag{3.6}$$

$$q_{i\theta_k\theta_j}(\mathbf{z}_i; \boldsymbol{\theta}) = \Delta_f(\|\mathbf{T}\|^2)\left(\frac{\partial\,\|\mathbf{T}_i\|^2}{\partial\theta_k}\right)\left(\frac{\partial\,\|\mathbf{T}_i\|^2}{\partial\theta_j}\right) + W_f(\|\mathbf{T}_i\|^2)\left(\frac{\partial^2\,\|\mathbf{T}_i\|^2}{\partial\theta_k\,\partial\theta_j}\right), \tag{3.7}$$

and

$$q_{i\theta_k, \theta_j}(\mathbf{z}_i, \boldsymbol{\theta}) = W_f^2(\|\mathbf{T}_i\|^2)\left(\frac{\partial\,\|\mathbf{T}_i\|^2}{\partial\theta_k}\right)\left(\frac{\partial\,\|\mathbf{T}_i\|^2}{\partial\theta_j}\right), \tag{3.8}$$

$j, k=1, ..., 2q$, where $\theta_j = \alpha_j$, $j=1, ..., q$, $\theta_{j+q} = \beta_j$, $j=1, ..., q$. Moreover, straightforward but lengthy algebraic manipulations show that

$$\frac{\partial\,\|\mathbf{T}_i\|^2}{\partial\alpha_j} = -2(\mathbf{d}_j'\mathbf{T}_i),$$

$$\frac{\partial\,\|\mathbf{T}_i\|^2}{\partial\beta_j} = -2(\mathbf{d}_j'\mathbf{T}_i)\,x_{i0} - \frac{2}{\|\mathbf{b}\|}(\mathbf{d}_j'\mathbf{T}_i)\,r_i,$$

$$\frac{\partial^2\,\|\mathbf{T}_i\|^2}{\partial\alpha_j\,\partial\alpha_k} = 2\mathbf{d}_j'\mathbf{B}_2\mathbf{d}_k,$$

$$\frac{\partial \|\mathbf{T}_i\|^2}{\partial \alpha_j \partial \beta_k} = \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \alpha_j \partial \alpha_k} x_{i0} + \frac{2}{\|\mathbf{b}\|} (\mathbf{d}_j' \mathbf{B}_2 \mathbf{d}_k) r_i + \frac{2}{\|\mathbf{b}\|} (\mathbf{d}_j' \mathbf{b}_1) \mathbf{d}_k' \mathbf{T}_i,$$

$$\frac{\partial \|\mathbf{T}_i\|^2}{\partial \beta_j \partial \beta_k} = \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \alpha_j \partial \alpha_k} x_{i0}^2 + \frac{4x_{i0}}{\|\mathbf{b}\|} (\mathbf{d}_j' \mathbf{B}_2 \mathbf{d}_k) r_i + \frac{2x_{i0}}{\|\mathbf{b}\|} \mathbf{b}_1' \mathbf{D}(j, k) \mathbf{T}_i$$

$$+ \frac{2}{\|\mathbf{b}\|^2} \mathbf{b}_1' \mathbf{D}(j, k) \mathbf{T}_i r_i - \frac{2}{\|\mathbf{b}\|^2} (\mathbf{T}_i' \mathbf{D}_{jk} \mathbf{T}_i) + \frac{2}{\|\mathbf{b}\|^2} (\mathbf{d}_j' \mathbf{B}_2 \mathbf{d}_k) r_i^2,$$

with $\mathbf{D}_{jk} = \mathbf{d}_j \mathbf{d}_k'$, $\mathbf{D}(j, k) = \mathbf{D}_{jk} + \mathbf{D}_{kj}$, and $\mathbf{d}_j = (\boldsymbol{\delta}_j', 0)' \in \mathcal{R}^p$, $p = q+1$, where $\boldsymbol{\delta}_j = (0, ..., 0, 1, 0, ..., 0)' \in \mathcal{R}^q$, $j = 1, ..., q$, that is, a q-dimensional vector of zeroes with one in the jth position.

To prove the main results, the following assumptions are required:

(C.3) There are positive constants a_f and b_f such that

$$|W_f(u)| \leqslant a_f, \qquad |W_{f'}(u)| \leqslant b_j, \qquad u \geqslant 0;$$

(C.4) For each $\bar{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^0$ (the interior of $\boldsymbol{\Theta}$), there exists $\delta > 0$ and functions $d_i(\mathbf{z}_i)$ and $d_{ikj}(\mathbf{z}_i)$ such that

$$|h_i(\mathbf{z}_i, \boldsymbol{\theta})| \leqslant d_i(\mathbf{z}_i)$$

and

$$|q_{i\theta_k\theta_j}(\mathbf{z}_i; \boldsymbol{\theta})| \leqslant d_{ikj}(\mathbf{z}_i)$$

for all $\boldsymbol{\theta} \in V(\bar{\boldsymbol{\theta}}, \delta) = \{\boldsymbol{\theta}; \|\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}\| < \delta\} \subset \boldsymbol{\Theta}^0$, and

$$\limsup n^{-1} \sum_{i=1}^{n} E_0[d_i^2(\mathbf{Z}_i)] < \infty, \qquad \limsup n^{-1} \sum_{i=1}^{n} E_0[d_{ikj}^2(\mathbf{Z}_i)] < \infty,$$

$j, k = 1, ..., 1q$;

(C.5) there exists $\gamma > 0$ such that

$$E_0[|W_f(\|\mathbf{e}_1\|^2) \|\mathbf{e}_1\| |u_1|^{2+\gamma}] < \infty;$$

(C.6) the sequence $\{x_i\}$, $i \geqslant 1$, is such that there exists $M > 0$ such that

$$\sup_{i \geqslant 1} |x_i| = M < \infty;$$

(C.7) given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\limsup n^{-1} \left| \sum_{i=1}^{n} E_0[\sup \{q_{i\theta_k\theta_l}(\mathbf{Z}_i; \boldsymbol{\theta}); \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\| < \delta\} - q_{i\theta_k\theta_l}(\mathbf{Z}_i; \boldsymbol{\theta}_0)] \right| < \varepsilon,$$

$k, l = 1, ..., 2q$. The same is true when sup is replaced by inf.

The lemma presented next will be usefull in proving the main results in the paper. The notation $\otimes$ indicates the Kronecker product.

LEMMA 3.2. *Consider the functional model* (1.2)–(1.4). *Under conditions* (C.1)–(C.2), *the matrices* $E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)]$ *and* $\mathbf{V}_n(\boldsymbol{\theta}_0)$ *defined in* (2.6) *and* (2.9), *respectively, are such that*

$$E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)] = 2C_f \begin{pmatrix} 1 & \bar{x}_0 \\ \bar{x}_0 & S_x^0 + \dfrac{1}{\|\mathbf{b}_0\|^2} \dfrac{B_f}{C_f} \end{pmatrix} \otimes \mathbf{B}_{30}$$

and

$$\mathbf{V}_n(\boldsymbol{\theta}_0) = \frac{4}{q} E_0[W_f^2(\|\mathbf{e}_1\|^2) \|\mathbf{e}_1\|^2] \begin{pmatrix} 1 & \bar{x}_0 \\ \bar{x}_0 & S_x^0 + \dfrac{1}{\|\mathbf{b}_0\|^2} A_f \end{pmatrix} \otimes \mathbf{B}_{30},$$

where

$$\bar{x}_0 = \frac{1}{n} \sum_{i=1}^{n} x_{i0}, \qquad S_x^0 = \frac{1}{n} \sum_{i=1}^{n} x_{i0}^2, \qquad \mathbf{B}_{30} = \mathbf{I}_q - \frac{\boldsymbol{\beta}_0 \boldsymbol{\beta}_0'}{\|\mathbf{b}_0\|^2},$$

$$A_f = \frac{E_0[W_f^2(\|\mathbf{e}_1\|^2)\, a(\|\mathbf{e}_1\|^2)\, \|\mathbf{e}_1\|^2]}{E_0[W_f^2(\|\mathbf{e}_1\|^2)\, \|\mathbf{e}_1\|^2]},$$

$$B_f = E_0\left[W_f(\|\mathbf{e}_1\|^2) \left(a(\|\mathbf{e}_1\|^2) - \frac{1}{q} \|\mathbf{e}_1\|^2 \right) \right] + \frac{2}{q} E_0[\Delta_f(\|\mathbf{e}_1\|^2)\, a(\|\mathbf{e}_1\|^2)\, \|\mathbf{e}_1\|^2],$$

and

$$C_f = E_0[W_f(\|\mathbf{e}_1\|^2)] + \frac{2}{q} E_0[\Delta_f(\|\mathbf{e}_1\|^2)\, \|\mathbf{e}_1\|^2],$$

with $a(\|\mathbf{e}_1\|^2) = E_0[u_1^2 \mid \mathbf{e}_1]$ *and* $(\mathbf{e}_1', u_1)' \sim El_p(\mathbf{0}, \mathbf{I}_p; f)$.

Proof. From (3.7) and using the fact that (see Lemma A.1 and (3.4))

$$\begin{pmatrix} \mathbf{T}_{i0} \\ r_{i0} \end{pmatrix} \overset{d}{=} \begin{pmatrix} \boldsymbol{\Gamma}'\mathbf{Q}_1'\mathbf{e}_1 \\ \mathbf{b}_{10}'\boldsymbol{\Gamma}'\mathbf{Q}_2'u_1 \end{pmatrix},$$

it follows that

$$E_0[q_{i\alpha_k\alpha_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)]$$
$$= 2\{2E_0[\Delta_f(\|\mathbf{T}_{i0}\|^2)(T'_{i0}\mathbf{D}_{kj}\mathbf{T}_{i0})] + (\mathbf{d}_j'\mathbf{B}_2\mathbf{d}_k)\, E_0[W_f(\|\mathbf{T}_{i0}\|^2)]\}.$$

Moreover, since the functions W_f and Δ_f are continuous functions, it follows from Lemma A.1 in the Appendix that

$$\Delta_f(\|\mathbf{T}_{i0}\|^2)(\mathbf{T}'_{i0}\mathbf{D}_{jk}\mathbf{T}_{i0}) \overset{d}{=} \Delta_f(\|\mathbf{e}_1\|^2)(\mathbf{e}_1'\mathbf{Q}_1\boldsymbol{\Gamma}\mathbf{D}_{jk}\boldsymbol{\Gamma}'\mathbf{Q}_1'\mathbf{e}_1)$$

and

$$W_f(\|\mathbf{T}_{i0}\|^2) \overset{d}{=} W_f(\|\mathbf{e}_1\|^2).$$

Since $\|\mathbf{e}_1\|^2$ is independent of $\mathbf{u}^{(q)} \overset{d}{=} \mathbf{e}_1/\|\mathbf{e}_1\|$, then from Lemma A.2 and properties of the elliptical distributions, it follows that

$$E_0[q_{i\alpha_k\alpha_j}(\mathbf{Z}_i;\boldsymbol{\theta})]$$
$$= 2(\mathbf{d}_j'\mathbf{B}_{20}\mathbf{d}_k)\left\{E_0[W_f(\|\mathbf{e}_1\|^2)] + \frac{2}{q}E_0[\Delta_f(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2]\right\}.$$

In a similar way, using the fact that $E[r_{i0}\,|\,\mathbf{T}_{i0}] = 0$, since $(r_{i0}, \mathbf{T}'_{i0})' \overset{iid}{\sim} El_{p+1}(\mathbf{0}, \mathbf{B}_0)$, where

$$\mathbf{B}_0 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{20} \end{pmatrix},$$

it can be shown that

$$E_0[q_{i\alpha_k\beta_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] = x_{i0}E_0[q_{i\alpha_k\alpha_j}(\mathbf{Z}_j;\boldsymbol{\theta}_0)].$$

Now, from (3.7) and from the fact that $E[r_{i0}\,|\,\mathbf{T}_{i0}] = 0$, it follows that

$$E_0[q_{i\beta_k\beta_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)]$$
$$= x_{i0}^2E_0[q_{i\alpha_k\alpha_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] + \frac{2}{\|\mathbf{b}_0\|^2}\{2E_0[\Delta_f(\|\mathbf{T}_{i0}\|^2)(\mathbf{T}'_{i0}\mathbf{D}_{jk}\mathbf{T}_{i0})\, r_{i0}^2]$$
$$+ (\mathbf{d}_j'\mathbf{B}_{20}\mathbf{d}_k)\, E_0[W_f(\|\mathbf{T}_{i0}\|^2)\, r_{i0}^2] - E_0[W_f(\|\mathbf{T}_{i0}\|^2)(\mathbf{T}'_{i0}\mathbf{D}_{jk}\mathbf{T}_{i0})]\}.$$

The continuity of the functions W_f and Δ_f in conjunction with Lemma A.1 imply that

$$\Delta_f(\|\mathbf{T}_{i0}\|^2)(\mathbf{T}'_{i0}\mathbf{D}_{jk}\mathbf{T}_{i0})\, r_{i0}^2 \overset{d}{=} \Delta_f(\|\mathbf{e}_1\|^2)(\mathbf{e}'_1\mathbf{Q}_1\boldsymbol{\Gamma}\mathbf{D}_{jk}\boldsymbol{\Gamma}'\mathbf{Q}'_1\mathbf{e}_1)\, u_1^2,$$
$$W_f(\|\mathbf{T}_{i0}\|^2)\, r_{i0}^2 \overset{d}{=} W_f(\|\mathbf{e}_1\|^2)\, u_1^2,$$

and

$$W_f(\|\mathbf{T}_{i0}\|^2)(\mathbf{T}'_{i0}\mathbf{D}_{jk}\mathbf{T}_{i0}) \overset{d}{=} W_f(\|\mathbf{e}_1\|^2)(\mathbf{e}'_1\mathbf{Q}_1\boldsymbol{\Gamma}_1\mathbf{D}_{jk}\boldsymbol{\Gamma}'\mathbf{Q}'_1\mathbf{e}_1).$$

From the above results and Lemma A.3, it follows that

$$E_0[q_{i\beta_k\beta_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] = x_{i0}^2 E_0[q_{i\alpha_k\alpha_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] + \frac{2}{\|\mathbf{b}_0\|^2}(\mathbf{d}'_j\mathbf{B}_{20}\mathbf{d}_k)\, B_f.$$

Moreover, since $\mathbf{d}_j = (\boldsymbol{\delta}'_j, 0)'$, we have that

$$b_{jk}^0 = \mathbf{d}'_j\mathbf{B}_{20}\mathbf{d}_k = \boldsymbol{\delta}'_j\mathbf{B}_{30}\boldsymbol{\delta}_k,$$

which is the (j, k)th entry of the matrix

$$\mathbf{B}_{30} = \mathbf{I}_q = \frac{\boldsymbol{\beta}_0\boldsymbol{\beta}'_0}{\|\mathbf{b}_0\|^2}, \qquad j, k = 1, ..., q.$$

Thus, the matrix $E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)]$ follows from the above results and from (2.6).

Similarly, (3.8), Lemmas A.1 and A.2, and the fact that the distribution of $(r_{i0}, \mathbf{T}'_{i0})'$ is symmetric (in relation to the origin), imply that the entries of the matrix $\mathbf{V}_n(\boldsymbol{\theta}_0)$ defined in (2.9) are given by

$$E_0[q_{i\alpha_k,\,\alpha_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] = \frac{4}{q}\, b_{jk}^0 E_0[W_f^2(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2],$$
$$E_0[q_{i\alpha_k,\,\beta_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] = x_{i0} E_0[q_{i\alpha_k,\,\alpha_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)],$$

and

$$E_0[q_{i\beta_k,\,\beta_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] = x_{i0}^2 E_0[q_{i\alpha_k,\,\alpha_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] + \frac{4}{q\,\|\mathbf{b}_0\|^2}\, b_{jk}^0 E_0[W_f^2(\|\mathbf{e}_1\|^2)\, a(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2],$$

which are based on the fact that

$$E_0[q_{i\alpha_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] = E_0[q_{i\beta_j}(\mathbf{Z}_i;\boldsymbol{\theta}_0)] = 0. \tag{3.9}$$

It can be shown that $C_f < 0$ and $B_f = 0$ for the normal, Student-t, and generalized Student-t models and we conjecture that this is also valid for all distributions in the elliptical family which verify the required conditions. The main results of the paper are presented in the sequel. The first important result shows that $\boldsymbol{\theta}_1$ (defined in Section 2) coincides with the true value $\boldsymbol{\theta}_0$ under some special conditions.

THEOREM 3.1. *If $C_f < 0$, $B_f = 0$ and conditions* (C.1)–(C.3) *are satisfied, then $\boldsymbol{\theta}_0$ is a local maximum of the function $\bar{\psi}(\boldsymbol{\theta}) = n^{-1} E_0[\sum_{i=1}^n h_i(\mathbf{Z}_i; \boldsymbol{\theta})]$, where $h_i(\mathbf{z}_i; \boldsymbol{\theta}) = \log f(\|\mathbf{T}_i\|^2)$, $i = 1, ..., n$.*

Proof. For any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ it follows from (3.6) that

$$q_{i\alpha_j}(\mathbf{z}_i; \boldsymbol{\theta}) = -2W_f(\|\mathbf{T}_i\|^2)(\mathbf{d}_j'\mathbf{T}_i)$$

and

$$q_{i\beta_j}(\mathbf{z}_i; \boldsymbol{\theta}) = q_{i\alpha_j}(\mathbf{z}_i; \boldsymbol{\theta})\, x_{i0} - \frac{2}{\|\mathbf{b}\|} W_f(\|\mathbf{T}_i\|^2)(\mathbf{d}_j'\mathbf{T}_i)\, r_i, \qquad j = 1, ..., q.$$

For $\boldsymbol{\theta} \in \mathbf{V}(\boldsymbol{\theta}_0, \delta)$, $\delta > 0$, it follows that

$$\|\mathbf{z}_i - \mathbf{a} - \mathbf{b}x_{i0}\| \leqslant \|\mathbf{z}_i - \mathbf{a}_0 - \mathbf{b}_0 x_{i0}\| + \delta(1 + |x_{i0}|) = d_0(\mathbf{z}_i),$$

implying that

$$\|\mathbf{T}_i\| = \|\mathbf{B}_2(\mathbf{z}_i - \mathbf{a} - \mathbf{b}x_{i0})\| \leqslant \sqrt{q}\, d_0(\mathbf{z}_i)$$

and

$$|r_i| \leqslant d_0(\mathbf{z}_i).$$

Thus, condition (C.3) and the Cauchy–Schwarz inequality imply that

$$|q_{i\alpha_j}(\mathbf{z}_i; \boldsymbol{\theta})| \leqslant 2\sqrt{q}\, a_f\, d_0(\mathbf{z}_i), \qquad j = 1, ..., q,$$

and

$$|q_{i\beta_j}(\mathbf{z}_i; \boldsymbol{\theta})| \leqslant 2\sqrt{q}\, a_f(d_0(\mathbf{z}_i)\,|x_{i0}| + d_0^2(\mathbf{z}_i)), \qquad j = 1, ..., q.$$

Defining

$$\psi_i(\boldsymbol{\theta}) = E_0[h_i(\mathbf{Z}_i; \boldsymbol{\theta})],$$

the above inequalities, condition (C.2), and the dominated convergence theorem permit us to write

$$\frac{\partial \psi_i(\boldsymbol{\theta})}{\partial \theta_j} = E_0[q_{i\theta_j}(\mathbf{Z}_i; \boldsymbol{\theta})],$$

$j = 1, ..., 2q$. Thus, from (3.9), it follows that $\boldsymbol{\theta}_0$ is a critical point of the function $\psi_i(\boldsymbol{\theta})$, $i = 1, ..., n$, and, accordingly, a critical point of $\bar{\psi}(\boldsymbol{\theta})$. On the other hand, condition (C.3) and the Cauchy–Schwarz inequality imply that

$$|q_{i\alpha_k\alpha_j}(\mathbf{z}_i; \boldsymbol{\theta})| \leqslant 4qc_f d_0^2(\mathbf{z}_i) + 2qa_f,$$

$$|q_{i\alpha_k\beta_j}(\mathbf{z}_i; \boldsymbol{\theta})| \leqslant 4qc_f(d_0^2(\mathbf{z}_i)\, |x_{i0}| + d_0^3(\mathbf{z}_i)) + 2qa_f(|x_{i0}| + 2d_0(\mathbf{z}_i)),$$

and

$$|q_{i\beta_k\beta_j}(\mathbf{z}_i; \boldsymbol{\theta})| \leqslant 4qc_f(d_0(\mathbf{z}_i)\, |x_{i0}| + d_0^2(\mathbf{z}_i))^2 + 2qa_f(x_{i0}^2 + 4\, |x_{i0}|\, d_0(\mathbf{z}_i) + 4d_0^2(\mathbf{z}_i)),$$

$j, k = 1, ..., q$, where $c_f = a_f(a_f + b_f)$. Thus, condition (C.2) and the dominated convergence theorem imply that

$$\frac{\partial^2 \psi_i(\boldsymbol{\theta})}{\partial \theta_k\, \partial \theta_j} = E_0[q_{i\theta_k\theta_j}(\mathbf{Z}_i; \boldsymbol{\theta})],$$

$j, k = 1, ..., 2q$, with $\psi_i \in \mathscr{C}^{(2)}$ (see condition (C.1)), $i = 1, ..., n$. Moreover, the matrix of second derivatives of the function $\psi_i(\boldsymbol{\theta})$ at $\boldsymbol{\theta}_0$ is given by

$$2C_f \begin{pmatrix} 1 & x_{i0} \\ x_{i0} & x_{i0}^2 + \dfrac{1}{\|\mathbf{b}\|^2} \dfrac{B_f}{C_f} \end{pmatrix} \otimes \mathbf{B}_{30},$$

where $\mathbf{B}_{30}$ is as in Lemma 3.2. Accordingly, the matrix of second derivatives of $\bar{\psi}(\boldsymbol{\theta})$, namely, $E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)]$, is as given in Lemma 3.2 with $B_f = 0$. In the following it is shown that the eigenvalues of the matrix $E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)]$ are all negative and consequently $\boldsymbol{\theta}_0$ is a local maximum of the function $\bar{\psi}(\boldsymbol{\theta})$. As such, the eigenvalues of the matrix

$$\mathbf{P}_0 = \begin{pmatrix} 1 & \bar{x}_0 \\ \bar{x}_0 & S_x^0 \end{pmatrix}$$

are given by $\lambda_i = 1 + S_x^0 \mp \sqrt{(1 + S_x^0)^2 - 4S_{xx}^0}$, $i = 1, 2$, which are all positives, with $S_{xx}^0 = S_x^0 - \bar{x}_0^2$. On the other hand, it follows that

$$|\mathbf{B}_{30} - \delta \mathbf{I}_q| = \left| \mathbf{I}_q - \frac{\boldsymbol{\beta}_0 \boldsymbol{\beta}_0'}{\|\mathbf{b}_0\|^2} - \delta \mathbf{I}_q \right|$$

$$= \left| (1-\delta)\, \mathbf{I}_q - \frac{\boldsymbol{\beta}_0 \boldsymbol{\beta}_0'}{\|\mathbf{b}_0\|^2} \right| = \left(\frac{\|\boldsymbol{\beta}_0\|^2}{\|\mathbf{b}_0\|^2} \right)^q |\gamma \mathbf{I}_q - \mathbf{B}_0|,$$

where

$$\gamma = \frac{\|\mathbf{b}_0\|^2}{\|\boldsymbol{\beta}_0\|^2} (1-\delta), \qquad \mathbf{B}_0 = \frac{\boldsymbol{\beta}_0 \boldsymbol{\beta}_0'}{\|\boldsymbol{\beta}_0\|^2}. \tag{3.10}$$

Since $\mathbf{B}_0$ is a symmetric and idempotent matrix with rank one, it follows that the eigenvalues of the matrix $\mathbf{B}_0$ are $\gamma_1 = 1$ and $\gamma_2 = \cdots = \gamma_q = 0$, so that the eigenvalues of the matrix $\mathbf{B}_{30}$ are given by $\delta_1 = 1/\|b_0\|^2$ and $\delta_2 = \cdots = \delta_q = 1$. Thus, the eigenvalues of the matrix $E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)] = 2C_f(\mathbf{P}_0 \otimes \mathbf{B}_{30})$ are all negative, since $C_f < 0$.

The result presented next shows that the maximum likelihood estimator $\tilde{\boldsymbol{\theta}}_n$ obtained by solving simultaneously the $2q$ equations

$$\sum_{i=1}^{n} q_{i\alpha_j}(\mathbf{z}_i; \boldsymbol{\theta}) = 0 \tag{3.11}$$

and

$$\sum_{i=1}^{n} q_{i\beta_j}(\mathbf{z}_i; \boldsymbol{\theta}) = 0, \tag{3.12}$$

$j = 1, ..., q$, are consistent and asymptotically normal.

THEOREM 3.2. *Consider the model given in* (1.2)–(1.4) *satisfying conditions* (C.1)–(C.7) *with* $C_f < 0$ *and* $B_f = 0$. *Then, the maximum likelihood estimator* $\tilde{\boldsymbol{\theta}}_n = (\tilde{\boldsymbol{\alpha}}_n', \tilde{\boldsymbol{\beta}}_n')'$ *of* $\boldsymbol{\theta}_0$, *obtained by solving Eqs.* (3.11) *and* (3.12) *is such that* $\tilde{\boldsymbol{\theta}}_n \xrightarrow{P} \boldsymbol{\theta}_0$ *and is asymptotically normally distributed with mean vector* $\boldsymbol{\theta}_0$ *and covariance matrix given by*

$$\boldsymbol{\Sigma}_n = \frac{E_0[W_f^2(\|\mathbf{e}_1\|^2)\, \|\mathbf{e}_1\|^2]}{qC_f^2(S_{xx}^0)^2} \begin{pmatrix} \Delta(S_{xx}^0)^2 + K\bar{x}_0^2 & -K\bar{x}_0 \\ -K\bar{x}_0 & K \end{pmatrix} \otimes \mathbf{B}_{40},$$

where A_f *is as in Lemma* 3.2, $\Delta = \|\mathbf{b}_0\|^2$, $K = \Delta S_{xx}^0 + A_f$, *and*

$$\mathbf{B}_{40} = \frac{1}{\|\mathbf{b}_0\|^2} (\mathbf{I}_q + \boldsymbol{\beta}_0 \boldsymbol{\beta}_0').$$

Proof. Under the elliptical functional model satisfying conditions (C.1)–(C.7), it can be shown, by using the Cauchy–Schwarz inequality that the regularity conditions defined in Mak (1982, Section 2), are satisfied. Thus, it follows that the maximum likelihood estimator which follows by solving Eqs. (3.11) and (3.12) is a consistent estimator of $\boldsymbol{\theta}_0$, asymptotically normally distributed with covariance matrix following from (2.7) and given by

$$\boldsymbol{\Sigma}_n = (E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)])^{-1}\,\mathbf{V}_n(\boldsymbol{\theta}_0)(E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)])^{-1},$$

where, from Lemma 3.2, under the assumption that $B_f = 0$,

$$\mathbf{V}_n(\boldsymbol{\theta}_0) = \frac{4}{q}\,E_0[\,W_f^2(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2]\begin{pmatrix} 1 & \bar{x}_0 \\ \bar{x}_0 & S_x^0 + \dfrac{1}{\|\mathbf{b}_0\|^2}\,A_f \end{pmatrix} \otimes \mathbf{B}_{30}$$

and

$$(E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)])^{-1} = \frac{1}{2C_f S_{xx}^0}\begin{pmatrix} S_x^0 & -\bar{x}_0 \\ -\bar{x}_0 & 1 \end{pmatrix} \otimes \mathbf{B}_{30}^{-1}.$$

In the case of $q = 1$, the generalized least squares estimator (Sprent, 1966; Gleser, 1985), is obtained by minimizing the function $Q_G(\alpha, \beta, \mathbf{x}) = \sum_{i=1}^n \boldsymbol{\varepsilon}_i' \boldsymbol{\varepsilon}_i$, where $\boldsymbol{\varepsilon}_i = \mathbf{Z}_i - \mathbf{a} - \mathbf{b}x_i$, $\mathbf{x} = (x_1, ..., x_n)'$, and is given by

$$\hat{\beta}_{GLS} = \frac{S_{YY} - S_{XX} + \sqrt{(S_{YY} - S_{XX})^2 + 4S_{XY}^2}}{2S_{XU}},$$

where $S_{YY} = \sum_{i=1}^n (Y_i - \bar{Y})^2/n$, $S_{XY} = \sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})/n$ and $S_{XX} = \sum_{i=1}^n (X_i - \bar{X})^2/n$. The following result presents the asymptotic relative efficiency $\mathrm{e}_{\hat{\beta}_{GLS}, \tilde{\beta}_n}$ of the generalized least squares estimator with respect to the estimator $\tilde{\beta}_n$ which is obtained by solving equations (3.11)–(3.12) for the case of $q = 1$. It depends on the condition:

(C.8) The sequence $\{x_i\}_{i \geqslant 1}$ is such that $\bar{x} = \sum_{i=1}^n x_i/n \to \mu$ and $\sum_{i=1}^n (x_i - \bar{x})^2/n \to \sigma_{xx}^*$.

Corollary 3.1. *Under the assumptions considered in Theorem* 3.2, *condition* (C.8), *and* $q = 1$, *it follows that*

$$\mathrm{e}_{\hat{\beta}_{GLS}, \tilde{\beta}} = \frac{E_0[\,W_f^2(e_1^2)\,e_1^2]}{C_f^2 \delta}\left(\frac{\Delta\sigma_{xx}^* + A_f}{\Delta\sigma_{xx}^* + \delta(\kappa + 1)}\right),$$

where δ *and* κ *are as in* (A.1) *and* (A.2), *respectively.*

Proof. From Theorem 3.2 in Arellano-Valle *et al.* (1996), it follows, as $n \to \infty$, that

$$\sqrt{n}(\hat{\beta}_{GLS} - \beta_0) \xrightarrow{d} N(0, \sigma^2_{GLS}),$$

where

$$\sigma^2_{GLS} = \frac{\delta}{(\sigma^*_{xx})^2} (\Delta \sigma^*_{xx} + \delta(\kappa + 1)). \tag{3.13}$$

Moreover, from Theorem 3.2 and the Slutsky theorem, it follows that

$$\sqrt{n}(\tilde{\beta}_n - \beta_0) \xrightarrow{d} N(0, \sigma^2_M),$$

where

$$\sigma^2_M = \frac{E_0[W_f(e_1^2)\, e_1^2]}{C_f^2 (\sigma^*_{xx})^2} (\Delta \sigma^*_{xx} + A_f). \tag{3.14}$$

Thus, the result follows from (3.13) and (3.14) and from the fact that $\mathrm{e}_{\hat{\beta}_{GLS}, \tilde{\beta}_n} = \sigma^2_M / \sigma^2_{GLS}$.

4. THE MULTIUNIVARIATE STUDENT-t FUNCTIONAL MODEL

In this section we consider that

$$\boldsymbol{\varepsilon}_i \overset{iid}{\sim} t_p(\mathbf{0}, \mathbf{I}_p; \nu), \tag{4.1}$$

$i = 1, \dots, n$, with the density function given in (1.6). After some algebraic manipulations (see Section A.1 in the Appendix, with $\lambda = \nu$, $W_f = W_p$), it follows that

$$W_f(\|\mathbf{e}_1\|^2) = -\frac{(\nu + p)}{2} (\nu + \|\mathbf{e}_1\|^2)^{-1}, \qquad a(\|\mathbf{e}_1\|^2) = \frac{\nu + \|\mathbf{e}_1\|^2}{\nu + q - 2},$$

and

$$\Delta_f(\|\mathbf{e}_1\|^2) = \frac{\nu + p}{2} (\nu + \|\mathbf{e}_1\|^2)^{-2},$$

where $(\mathbf{e}_1', u_1)' \sim t_p(\mathbf{0}, \mathbf{I}_p; \nu)$, $p = q + 1$, with $\mathbf{e}_1$ the q-dimensional vector as defined in (1.2). Moreover, it can be shown that

$$E_0[W_f(\|\mathbf{e}_1\|^2)] = -\frac{1}{2}\frac{v+p}{v+q},$$

$$E_0[W_f(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2] = -\frac{q}{2}\frac{v+p}{v+q},$$

$$E_0[W_f^2(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2] = \frac{q}{4}\frac{(v+p)^2}{(v+q)(v+q+2)},$$

$$E_0[\Delta_f(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2] = \frac{q}{2}\frac{v+p}{(v+q)(v+q+2)},$$

$$E_0[W_f(\|\mathbf{e}_1\|^2)\,a(\|\mathbf{e}_1\|^2)] = -\frac{1}{2}\frac{v+p}{v+q-2},$$

$$E_0[W_f^2(\|\mathbf{e}_1\|^2)\,a(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2] = \frac{q}{4}\frac{(v+p)^2}{(v+q)(v+q-2)}$$

and

$$E_0[\Delta_f(\|\mathbf{e}_1\|^2)\,a(\|\mathbf{e}_1\|^2)\,\|\mathbf{e}_1\|^2] = \frac{q}{2}\frac{v+p}{(v+q)(v+q-2)},$$

from where it follows that

$$A_f = \frac{v+q+2}{v+q-2}, \qquad B_f = 0, \qquad C_f = -\frac{1}{2}\frac{v+p}{v+q+2}. \tag{4.2}$$

Thus, from Lemma 3.2, it follows that

$$\mathbf{V}_n(\boldsymbol{\theta}_0) = \frac{(v+p)^2}{(v+q)(v+q+2)}\begin{pmatrix} 1 & \bar{x}_0 \\ \bar{x}_0 & S_x^0 + \frac{1}{\|\mathbf{b}_0\|^2}A_f \end{pmatrix} \otimes \mathbf{B}_{30} \tag{4.3}$$

and

$$E_0[\mathbf{A}_n(\boldsymbol{\theta}_0)] = -\frac{v+p}{v+q+2}\begin{pmatrix} 1 & \bar{x}_0 \\ \bar{x}_0 & S_x^0 \end{pmatrix} \otimes \mathbf{B}_{30}, \tag{4.4}$$

where $\mathbf{B}_{30}$ is as given in Lemma 3.2. It can also be shown that $B_f = 0$ and $C_f < 0$ in the case where $\boldsymbol{\varepsilon}_i \overset{iid}{\sim} t_p(\mathbf{0}; \mathbf{I}_p; \lambda, v)$, that is, $\boldsymbol{\varepsilon}_i$ follows the generalized Student-t distribution (see (A.3) in the Appendix), $i = 1, ..., n$.

After some algebraic manipulations, it can be shown that the maximum likelihood estimator of $\boldsymbol{\theta} = (\boldsymbol{\alpha}', \boldsymbol{\beta}')'$ is obtained by interactively solving equations (3.1)–(3.12), which in the case of a Student-t functional model can be written as

$$\sum_{i=1}^{n} \frac{\mathbf{d}_j' \mathbf{T}_i}{v + \|\mathbf{T}_i\|^2} \begin{pmatrix} 1 \\ x_{i0} + \dfrac{r_i}{\|\mathbf{b}\|} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.5}$$

since

$$q_{i\alpha_j}(\mathbf{z}_i; \boldsymbol{\theta}) = (v + p)(v + \|\mathbf{T}_i\|^2)^{-1} (\mathbf{d}_j' \mathbf{T}_i)$$

and

$$q_{i\beta_j}(\mathbf{z}_i; \boldsymbol{\theta}) = (v + p)(v + \|\mathbf{T}_i\|^2)^{-1} \mathbf{d}_j' \mathbf{T}_i \left(x_{i0} \frac{r_i}{\|\mathbf{b}\|} \right),$$

where r_i and $\mathbf{T}_i$ are as in (3.2) and (3.3), respectively. Bolfarine and Arellano-Valle (1994) consider the EM algorithm for obtaining the maximum likelihood estimator of $\boldsymbol{\theta}$ under the above Student-t model. As consequence of (4.3) and (4.4), we have the following result.

THEOREM 4.1. *Consider the model defined by* (1.3) *and* (4.1) *and with condition* (C.6) *satisfied. Then, for* $v > 4$, *the maximum likelihood estimator* $\tilde{\boldsymbol{\theta}}_n = (\tilde{\boldsymbol{\alpha}}_n', \tilde{\boldsymbol{\beta}}_n')'$ *of* $\boldsymbol{\theta}_0 = (\boldsymbol{\alpha}_0', \boldsymbol{\beta}_0')'$, *which is obtained by solving Eq.* (4.5), *is consistent and asymptotically normal with mean* $\boldsymbol{\theta}_0$ *and covariance matrix given by*

$$\boldsymbol{\Sigma}_n = \frac{v + q + 2}{v + q} \frac{1}{(S_{xx}^0)^2} \begin{pmatrix} \Delta (S_{xx}^0)^2 + K \bar{x}_0^2 & -K\bar{x}_0 \\ -K\bar{x}_0 & K \end{pmatrix} \otimes \mathbf{B}_{40},$$

where $\Delta = \|\mathbf{b}_0\|^2$, $K = \Delta S_{xx}^0 + A_f$, A_f *as defined in* (4.2), *and*

$$\mathbf{B}_{40} = \frac{1}{\|\mathbf{b}_0\|^2} (\mathbf{I}_q + \boldsymbol{\beta}_0 \boldsymbol{\beta}_0').$$

Proof. Ordinary algebraic manipulations show that the conditions (C.1)–(C.7) are satisfied for the Student-t model defined in (1.6) (see also (A.3) in the Appendix). Thus, the result follows from (4.3), (4.4), and Theorem 3.2.

COROLLARY 4.2. *Under the assumption given in Theorem* 4.1 *and condition* (C.8) *it follows that*

$$\mathrm{e}_{\hat{\beta}_{GLS},\tilde{\beta}_n} = \frac{(v-1)(v+3)}{v(v+1)} \left(\frac{\Delta\sigma^*_{xx} + \dfrac{v}{v-4}}{\Delta\sigma^*_{xx} + \dfrac{v+3}{v-1}} \right), \qquad v > 4, \tag{4.6}$$

which implies that $\tilde{\beta}_n$ *is more efficient that* $\hat{\beta}_{GLS}$.

Proof. The proof of (4.6) follows directly from Corollary 3.1, (4.2), by taking $\kappa + 1 = (v-2)/(v-4)$ and $\delta = v/(v-2)$. Moreover, notice that

$$\frac{(v-2)(v+3)}{v(v+1)} \leqslant 1, \qquad \frac{v+3}{v-1} \leqslant \frac{v}{v-4},$$

from where it follows that $\mathrm{e}_{\hat{\beta}_{GLS},\tilde{\beta}_n} \leqslant 1$, which concludes the proof.

Notice from Corollary 4.1 that $\mathrm{e}_{\hat{\beta}_{GLS},\tilde{\beta}_n}$ can be very low for v close to four.

5. FINAL CONCLUSIONS

The present paper considers maximum likelihood estimation in elliptical functional measurement error models. The multiunivariate model often used in comparative calibration is assumed. A particular important case of this model is the simple regression model with measurement errors (Fuller, 1987). The main limitation of the approach is that the dispersion matrix of the error vector $\boldsymbol{\varepsilon}_i$ in (1.3) is assumed to be known. Indeed, it is an open problem to extend the results of the paper to the situation where this dispersion matrix is unknown. An special situation not also solved and often considered in the literature (Bolfarine and Galea-Rojas, 1995) assumes that $\mathrm{Var}[\boldsymbol{\varepsilon}_i] = \sigma^2 \mathbf{I}_p$, where $\mathbf{I}_p$ is the identity matrix of dimension p and σ^2 is unknown. One way to counter this problem would be to consider a pseudo-likelihood approach (Gong and Samaniego, 1981; Kano *et al.*, 1993), where the unknown σ^2 is replaced by a consistent estimator. This approach may lead to some efficiency loss in the estimation of $\boldsymbol{\beta}$. The approach developed in the paper to study the multiunivariate model (1.3) can be extended to a multivariate set up by considering a s-dimensional vector $\mathbf{x}_i$ in place of the scalar x_i and a matrix $\mathbf{B}$ of dimension $q \times s$ in place of the vector $\boldsymbol{\beta}$. However, such an extension is hardly straightforward. A subject of further study would be to consider the possibility that the adopted elliptical model is not correctly specified. One such study developed in Kano *et al.* (1993) treats the case of ordinary elliptical models.

APPENDIX

In this appendix, properties of the elliptical distributions are presented, which are used in the proofs of the main results of the paper. Some of the results presented seem to be extensions of existing results in the literature. Let $\mathbf{X} \sim El_p(\mathbf{0}, \mathbf{I}_p)$ to denote the fact that the vector $\mathbf{X}$ is distributed according to the spherical distribution, meaning that $\mathbf{X} \stackrel{d}{=} \mathbf{\Gamma X}$ for all $\mathbf{\Gamma} \in \mathbf{O}_p = \{\mathbf{A}(p \times p); \mathbf{AA}' = \mathbf{I}_p\}$, namely, the group of p-dimensional orthogonal matrices. As considered in the Introduction, we also use the notation $El_p(\mathbf{0}, \mathbf{I}_p; f)$ $(El_p(\mathbf{0}, \mathbf{I}_o; \phi))$, to indicate situations where $\mathbf{X}$ has density function $f(\mathbf{t}'\mathbf{t})$ (characteristic function $\phi(\mathbf{tt}')$), $\mathbf{t} \in \mathscr{R}^p$. Moreover, if $\mathbf{X} \sim El_p(\mathbf{0}, \mathbf{I}_p; \phi)$, then $E[\mathbf{X}] = \mathbf{0}$ and $\operatorname{Cov}[\mathbf{X}] = \delta \mathbf{I}_p$, whenever existing where

$$\delta = -2\phi'(0), \qquad \text{with} \quad \phi'(0) = \left.\frac{d\phi(u)}{du}\right|_{u=0}, \tag{A.1}$$

and the kurthosis coefficient, κ, defined by

$$\kappa + 1 = \frac{\phi''(0)}{(\phi'(0))^2} \qquad \text{with} \quad \phi''(0) = \left.\frac{d\phi'(u)}{du}\right|_{u=0}. \tag{A.2}$$

LEMMA A.1. *Let* $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2')' \sim El_p(\mathbf{0}, \mathbf{I}_p)$, *where* $p_1 + p_2 = p$ *and* p_i *is the dimension of* $\mathbf{X}_i$, $i = 1, 2$. *Let* $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2')'$ *be the* $2p \times$ *one-dimensional random vector defined by* $\mathbf{Y}_i = \mathbf{A}_i \mathbf{X}$, $i = 1, 2$, *where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are symmetric matrices such that* $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{I}_p$ *and* $\operatorname{rank}(\mathbf{A}_i) = \operatorname{tr}(\mathbf{A}_i) = p_i$, $i = 1, 2$, *so that* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are idempotent orthogonal matrices. Then,*

(i) $\mathbf{Y} \sim El_{2p}(\mathbf{0}, \mathbf{A})$, *where*

$$\mathbf{A} = \operatorname{bdiag}(\mathbf{A}_1, \mathbf{A}_2) = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix},$$

and "bdiag" *denotes a block diagonal matrix*;

(ii) *There exists* $\mathbf{\Gamma} \in \mathbf{O}_p$ *such that*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \mathbf{\Gamma}'\mathbf{Q}_1'\mathbf{X}_1 \\ \mathbf{\Gamma}'\mathbf{Q}_2'\mathbf{X}_2 \end{pmatrix},$$

where $\mathbf{Q}_1 = [\mathbf{I}_{p_1} \ \mathbf{0}]$ $(p_1 \times p$*-dimensional*) *and* $\mathbf{Q}_2 = [\mathbf{0} \ \mathbf{I}_{p_2}]$ $(p_2 \times p$*-dimensional*), *with* $\mathbf{I}_{pi}$ *the* p_i*-dimensional identify matrix.*

Proof. (i) Let

$$\mathbf{C} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}$$

a $2p \times 2p$ dimensional matrix. Thus, $\mathbf{Y} = \mathbf{CX} \sim El_{2p}(\mathbf{0}, \mathbf{CC}')$, according to Fang *et al.* (1990), where $\mathbf{CC}' = \text{bdiag}(\mathbf{A}_1, \mathbf{A}_2)$, since $\mathbf{A}_i = \mathbf{A}_i^2$, $i = 1, 2$ and $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1 = \mathbf{0}$.

(ii) Since $\mathbf{A}_1$ and $\mathbf{A}_2 = \mathbf{I}_p - \mathbf{A}_1$ are idempotent matrices, there exists $\mathbf{\Gamma} \in \mathbf{O}_p$ such that $\mathbf{\Gamma A}_i\mathbf{\Gamma}' = \mathbf{Q}_i'\mathbf{Q}_i$ which implies that $\mathbf{A}_i = \mathbf{\Gamma}'\mathbf{Q}_i'\mathbf{Q}_i\mathbf{\Gamma}$, $i = 1, 2$, where $\mathbf{Q}_1 = [\mathbf{I}_{p_1} \;\; \mathbf{0}]$ and $\mathbf{Q}_2 = [\mathbf{0} \;\; \mathbf{I}_{p_2}]$. The previous results, in conjunction with $\mathbf{X} \stackrel{d}{=} \mathbf{\Gamma X}$, imply that

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1\mathbf{X} \\ \mathbf{A}_2\mathbf{X} \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}'\mathbf{Q}_1'\mathbf{Q}_1\mathbf{\Gamma X} \\ \mathbf{\Gamma}'\mathbf{Q}_2'\mathbf{Q}_2\mathbf{\Gamma X} \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \mathbf{\Gamma}'\mathbf{Q}_1'\mathbf{X}_1 \\ \mathbf{\Gamma}'\mathbf{Q}_2'\mathbf{X}_2 \end{pmatrix}.$$

An important consequence of the previous lemma is stated next.

COROLLARY A.1. *For any continuous function $g(\cdot)$ it follows that*

$$g(\mathbf{Y}_1, \mathbf{Y}_2) \stackrel{d}{=} g(\mathbf{\Gamma}'\mathbf{Q}_1'\mathbf{X}_1, \mathbf{\Gamma}'\mathbf{Q}_2'\mathbf{X}_2).$$

LEMMA A.2. *Let $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2')' \sim El_p(\mathbf{0}, \mathbf{I}_p)$ with $P[\mathbf{X} = \mathbf{0}] = 0$. Then,*

(i) $\mathbf{X}_1 \mid \mathbf{X}_2 \stackrel{d}{=} \mathbf{X}_1 \mid \|\mathbf{X}_2\|^2$, *and are elliptically distributed;*

(ii) *If $W = g(\|\mathbf{X}\|^2)$, $\mathbf{A}$ is a $p \times p$ symmetric matrix and $E[|W| \; \|\mathbf{X}\|^2] < \infty$, then*

$$E[W\mathbf{X}'\mathbf{AX}] = p^{-1}\,\text{tr}[\mathbf{A}]\, E[W \mid \|\mathbf{X}\|^2].$$

Proof. The proof of (i) can be seen in Fang *et al.* (1990). For proving (ii), it suffices to use the fact that $\mathbf{X} \stackrel{d}{=} R\mathbf{U}$, where $R \stackrel{d}{=} \|\mathbf{X}\|$ is independent of $\mathbf{U} \stackrel{d}{=} \mathbf{X}/\|\mathbf{X}\|$ and

$$E[\mathbf{U}'\mathbf{AU}] = p^{-1}\,\text{tr}[\mathbf{A}].$$

A.1. *The generalized Student-t distribution.* Let $\mathbf{X} \sim t_p(\mathbf{0}, \mathbf{I}_p; \lambda, \nu)$ an spherical p-variate Student-t distribution with density $f_p(\mathbf{x}'\mathbf{x})$, $\mathbf{x} \in \mathscr{R}^p$, where

$$f_p(u) = k(\nu, p)\, \lambda^{\nu/2}(\lambda + u)^{-(\nu+p)/2}, \qquad u \geqslant 0, \tag{A.3}$$

$\lambda, \nu > 0$, where $k(\nu, p)$ is as in (1.6). For $\lambda = \nu$, the usual Student-t distribution with ν degrees of freedom (see Arellano-Valle and Bolfarine, 1995) follows. Moreover, let $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2')' \sim t_p(\mathbf{0}, \mathbf{I}_p; \lambda, \nu)$, where $\mathbf{X}_i$ is p_i-dimensional, $i = 1, 2$, with $p_1 + p_2 = p$. We now define the function $W_p(u) = f_p'(u)/f_p(u)$, $u \geqslant 0$, with f_p the function given in (A.2). Some properties of $W_p(\|\mathbf{X}\|^2)$ with $\mathbf{X} \sim t(\mathbf{0}, \mathbf{I}_p; \lambda, \nu)$ are considered in Arellano-Valle and Bolfarine (1996). In the following we consider properties of the random variable $W_p(\|\mathbf{X}_1\|^2)$, where $\mathbf{X}_1$ is a q-dimensional subvector of $\mathbf{X}$, with $1 \leqslant q < p$, and f_q corresponding to the (marginal) density of $\mathbf{X}_1$.

LEMMA A.3. *Consider the generalized Student-t distribution with density given by* (A.3). *Then,*

(i) $W_p(u) = -v + p/2\,(\lambda + u)^{-1}$, $u \geqslant 0$;

(ii) $W_p(u) = (v + p/v + q)\, W_q(u)$, $u \geqslant 0$, $1 \leqslant q < p$; *and*

(iii) $W'_p(u) = 2(v+p)/(v+q)^2\, W_q^2(u)$, $u \geqslant 0$, $1 \leqslant q < p$, *where* W'_p *is the derivative of* W_p *and* $W_q(u) = f'_q(u)/f_q(u)$.

LEMMA A.4. *Let* $\mathbf{X} = (\mathbf{X}'_1, \mathbf{X}'_2)' \sim t_p(\mathbf{0}, \mathbf{I}_p; \lambda, v)$, *with* $\mathbf{X}_i$ *of dimension* p_i, $i = 1, 2$, *and* $p_1 + p_2 = p$. *Then,*

(i) $\mathbf{X}_2 \mid \mathbf{X}_1 \sim t_{p_2}(\mathbf{0}, \mathbf{I}_{p_2}; \lambda + \|\mathbf{X}_1\|^2, v + p_1)$;

(ii) $\mathrm{Var}[\mathbf{X}_2 \mid \mathbf{X}_1] = (\lambda + \|\mathbf{X}_1\|^2)/(v + p_1 - 2)\, \mathbf{I}_{p_2}$, $v + p_1 > 2$.

The proof of (ii) can be found in Arellano-Valle and Bolfarine (1995). The next result follows by using the previous lemmas, after straightforward but length algebraic manipulations.

LEMMA A.5. *Under the assumptions considered in Lemma* A.4, *with* $p_1 = q$ *and* $p_2 = 1$, *it follows that*

(1) $E[W_p(\|\mathbf{X}_1\|^2)] = -\frac{1}{2}(v(v+p)/2\lambda(v+q))$;

(2) $E[W_p(\|\mathbf{X}_1\|^2)\, \|\mathbf{X}_1\|^2] = -(q/2)(v+p)/(v+q)$,

(3) $E[W_p^2(\|\mathbf{X}_1\|^2)\, \|\mathbf{X}_1\|^2] = (q/4)(v(v+p)^2/\lambda(v+q)(v+q+2))$,

(4) $E[W'_p(\|\mathbf{X}_1\|^2)\, \|\mathbf{X}_1\|^2] = (q/2)(v(v+p)/\lambda(v+q)(v+q+2))$,

(5) $E[W_p(\|\mathbf{X}_1\|^2)\, X_2^2] = -\frac{1}{2}((v+p)/(v+q-2))$,

(6) $E[W_p^2(\|\mathbf{X}_1\|^2)\, \|\mathbf{X}_1\|^2\, X_2^2] = (q/4)((v+p)^2/(v+q)(v+q-2))$, *and*

(7) $E[W'_p(\|\mathbf{X}_1\|^2)\, \|\mathbf{X}_1\|^2\, X_2^2] = (q/2)((v+p)/(v+q)(v+q-2))$, *with* W'_p *as in Lemma* A.3.

ACKNOWLEDGMENTS

The authors acknowledge partial final support from Fondecyt 1960937/Chile and $\mathrm{CNP_q}$/Brasil. The authors acknowledge a referee for carefully reading and reviewing the paper.

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